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GLOBALLY CONTROLLABLE SYSTEMS OF RIGID BODIES†

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Natural Lagrangian systems simulating mechanisms with a kinematic tree structure are considered. Sufficient conditions are presented for global controllability, that is, for the possibility that an admissible input signal will steer the system, in a finite time, from any initial phase state to any specified state. For example, it is shown that a multi-segment pendulum in a horizontal plane (outside the gravitational field) is globally controllable by the sole action of a bounded external torque applied to the first segment.

Necessary and sufficient conditions for the controllability of linear systems are well known [1]. In the non-linear case, however, such properties can obviously only be evaluated by using the specific features of particular classes of objects.

Sufficient conditions have been obtained [2] for the global controllability (i.e. controllability in the entire phase space) of natural Lagrangian systems characterized by strong interaction of the segments, when partial dissipation (e.g. due to friction in a joint) could stabilize the entire system in a single stable equilibrium position.

Below the list of globally controllable objects will be extended to include certain systems of rigid bodies that admit of steady motions.

1. BASIC DEFINITIONS

Following [2], we consider natural Lagrangian systems, i.e. objects whose Lagrangian is symmetric with respect to time inversion $t \to -t$: $L(\mathbf{q}, \mathbf{q}^\cdot) = \frac{1}{2}\mathbf{q}^{\cdot T}\mathbf{A}(\mathbf{q})\mathbf{q}^\cdot - B(\mathbf{q})$ where $\mathbf{q} = (q_1, \ldots, q_n)^T$ is the configuration vector, $\mathbf{A}(\mathbf{q})$ is the (positive-definite) inertia matrix, and $B(\mathbf{q})$ is a scalar potential, which has a lower bound: $B(\mathbf{q}) \geqslant 0$, $B(\mathbf{0}) = 0$. The motion is described by the equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{q}^\cdot}\right) - \frac{\partial L}{\partial \mathbf{q}} = \mathbf{u}, \quad \mathbf{u} \in U \subset R^n \tag{1.1}$$

where $\mathbf{u} = (u_1, \ldots, u_n)^T$ is a vector of controls chosen from a prescribed bounded domain U containing $\mathbf{u} = \mathbf{0}$ as an interior point.

In free motion ($\mathbf{u} \equiv \mathbf{0}$) the energy integral of the system $E(\mathbf{q}, \mathbf{q}^\cdot) = 1/2\mathbf{q}^{\cdot T}\mathbf{A}(\mathbf{q})\mathbf{q}^\cdot + B(\mathbf{q})$ may be a periodic function of certain "angular" coordinates q_i ($i = 1, 2, \ldots, r$). They are chosen in a covering space $R^r \times R^{(n-r)}$ corresponding to the configuration space $M = T^r \times R^{(n-r)}$, where T^r is an r-dimensional torus. We will use the notation $\mathbf{q} \in M$. The phase space $TM = T^r \times R^{(2n-r)}$ is defined similarly, so that $(\mathbf{q}, \mathbf{q}^\cdot) \in TM$.

If the feedback $\mathbf{u} = \mathbf{u}(\mathbf{q}, \mathbf{q}^\cdot)$ is associated with a separatrix surface in TM, in which motion to a singular point takes place in infinite time, the surface is denoted by $\Omega(\mathbf{u}(\mathbf{q}, \mathbf{q}^\cdot)$. The set of equilibrium positions $\zeta_0 = \{(\mathbf{q}, \mathbf{q}^\cdot): \mathbf{q}^\cdot = 0, \partial B/\partial \mathbf{q} = \mathbf{0}, \mathbf{u} = \mathbf{0}\}$ is non-empty, the number of components of these manifolds is assumed to be finite.

For any scalar function $V(\mathbf{y})$, we let $Q_V = \{\mathbf{y}: \| \partial V/\partial \mathbf{y} \| = 0\}$ denote the set of critical points, $E_V = \{c: c = V(\mathbf{y})\}$ the value set and $H_c(V(\mathbf{y})) = \{y: V(\mathbf{y}) \leqslant c, c \in E_V\}$ the domains bounded by the level surfaces of the function.

The definitions of stabilizability and controllability will be used in the traditional sense [3].

Definition 1. System (1.1) is said to be stabilizable on $P \subset TM$ by inputs u_i ($i = 1, 2, \ldots, n$) if it can be steered from any point $(\mathbf{q}, \mathbf{q}^\cdot) \in P$ into an arbitrarily small neighbourhood of the state $(\mathbf{0}, \mathbf{0})$ on the

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assumption that $u_j \equiv 0$ $(j = m + 1, m + 2, \ldots, n)$.

A special case of this definition is stabilizability by a scalar (single) input u_i $(i = 1, 2, \ldots, n)$.

Definition 2. System (1.1) is said to be locally controllable by inputs u_i $(i = 1, 2, \ldots, m)$ in a neighbourhood of an equilibrium $(\mathbf{q}_0, \mathbf{0}) \in TM$ if $(\mathbf{q}_0, \mathbf{0})$ in a finite time by an admissible control, on the assumption that $u_k \equiv 0$ $(k = m + 1, m + 2, \ldots, n)$.

It is well known [3] that it is sufficient to observe local controllability in the linear approximation, applying the rank criterion of [1].

Definition 3. System (1.1) is said to be globally controllable in TM by inputs u_i $(i = 1, 2, \ldots, m)$ if it can be steered from an arbitrary state $(\mathbf{q}_1, \dot{\mathbf{q}}_1)$ to any prescribed state $(\mathbf{q}_2, \dot{\mathbf{q}}_2)$ in a finite time by admissible controls, on the assumption that $u_k \equiv 0$ $(k = m + 1, m + 2, \ldots, n)$.

The sufficient conditions formulated in [2] for global controllability of objects (1.1) when the number of controls is less than the number of degrees of freedom were based on the properties of stabilizability, local controllability of systems and their symmetry relative to time inversion. Stabilizability was proved by appeal to Lyapunov's direct method in stability theory. The well-known Barbashin–Krasovskii theorem [4] was extended to the case of a cylindrical phase space [5] by introducing a new concept—the connected Lyapunov function.

Definition 4. A single-valued function $V(\mathbf{y})$ $(\mathbf{y} \in T^k \times R^m)$ which is continuous together with its partial derivatives and positive-definite in Lyapunov's sense $(V(\mathbf{y}) \geqslant 0, V(\mathbf{y}) = 0 \Rightarrow \mathbf{y} = 0)$ is called a Lyapunov function on $T^k \times R^m$.

Definition 5. A Lyapunov function $V(\mathbf{y})$ $(\mathbf{y} \in T^k \times R^m)$ is called a connected Lyapunov function (CLF) in a domain $P \subset T^k \times R^m$ if every set $P \cap H_c\,(V(\mathbf{y}))$ is connected in P.

Various sufficient conditions for a Lyapunov function to be connected have been proved [5]. One of these may be formulated as follows: if the set $Q_V \backslash \mathbf{0}$ for a Lyapunov function $V(\mathbf{y})$ $(\mathbf{y} \in P)$, defined on a compact manifold $P \subset T^k \times R^m$, is the union of a finite number of isolated points, at each of which the matrix $\partial^2 V/\partial \mathbf{y}^2$ is not positive-definite, then $V(\mathbf{y})$ is a CLF on P. In other words, if the function $V(\mathbf{y})$ is not degenerate on a compact set P (that is, it is a Morse function [6]) and has only one local minimum (at the point $\mathbf{y} = \mathbf{0}$), then $V(\mathbf{y})$ is a *CLF* on P.

Example 1. A system of n rigid rods, joined together in sequence by cylindrical joints, is situated in a vertical plane (Fig. 1). All the angles φ_i $(i = 1, 2, \ldots, n)$ measure the deviation of the segments of the pendulum from the vertical axis, and the point of suspension is fixed. We will assume that the rods themselves are weightless, their masses being concentrated at the joints. The potential energy $B(\varphi) = \Sigma b_i(1 - \cos \varphi_i)$, where $b_i > 0$ $(i = 1, 2, \ldots, n)$, is a CLF on T^n.

We will present one result of [2] in a simplified form, confining our attention to the case of a one-dimensional control.

Proposition 1. Suppose that the potential $B(\mathbf{q})$ in system (1.1) is a *CLF* on M and that the sets $H_c(B(\mathbf{q}))$ are compact.

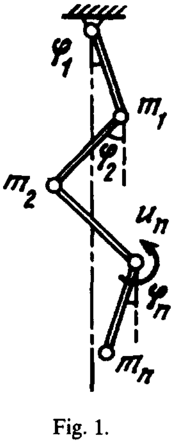

Fig. 1.

1. If the system in free motion ($u \equiv 0$) does not admit of the particular solution $q_j^{\cdot} \equiv 0$ (excluding equilibria), then it is stabilizable by input u_j ($j = 1, 2, \ldots, n$) on the manifold $TM \backslash \Omega(u_j)$.

2. Under these conditions, if the system is locally controllable in the neighbourhoods of all points $(\mathbf{q}_0, \mathbf{0}) \in \zeta_0$ by the same input u_j, then system (1.1) is globally controllable by the single control u_j.

The properties of the potential $B(\mathbf{q})$ enables us to analyse asymptotic stability in the large [5] by using the CLF $E(\mathbf{q}, \mathbf{q}^{\cdot})$—the total energy of the system. The role of the other conditions in the proposition is as follows:

1. the smooth feedback $u_j(\mathbf{q}, \mathbf{q}^{\cdot}) \in U$ (chosen so that sign u_j = −sign $\mathbf{q}_j^{\cdot}$) guarantees that a sufficiently small neighbourhood of one of the rest points $(\mathbf{q}_0, \mathbf{0}) \in \zeta_0$ is reached in a finite time;

2. if $\mathbf{q}_0 \neq \mathbf{0}$, local controllability enables one to "overcome jamming in the neighbourhood" (i.e. to complete the motion in the domain $TM \backslash \Omega(u_j)$ at a lower energy level) and to continue the stabilization of $u_j(\mathbf{q}, \mathbf{q}^{\cdot})$. But if $\mathbf{q} = \mathbf{0}$, local controllability guarantees that the point $(\mathbf{0}, \mathbf{0})$ will be reached in a finite time.

The following remark completes the logical chain of Proposition 1.

Remark. If an admissible control u_j will steer the Lagrangian system (1.1) in a finite time from any initial state $(\mathbf{q}_1, \mathbf{q}_1^{\cdot}) \in TM$ to the position $(\mathbf{0}, \mathbf{0})$, then the system is globally controllable by the input u_j ($j = 1, 2, \ldots, n$).

Indeed, Eqs (1.1) are invariant under the transformation $\mathbf{q}^{\cdot} \to -\mathbf{q}^{\cdot}$, $t \to -t$, any controllable process $u_j(t)$: $(\mathbf{q}_2, \mathbf{q}_2^{\cdot}) \to (\mathbf{0}, \mathbf{0})$ (as $t \to [0, T]$) can be associated in the same interval of t with a "symmetric" motion $u_j(T - t)$: $(\mathbf{0}, \mathbf{0}) \to (\mathbf{q}_2\, \mathbf{q}_2)$, and we can thus construct a stepwise transition $(\mathbf{q}_1, \mathbf{q}_1^{\cdot}) \to (\mathbf{0}, \mathbf{0}) \to (\mathbf{q}_2, \mathbf{q}_2^{\cdot})$.

For Example 1, the proposition implies that the n-segment pendulum is globally controllable in a vertical plane (with friction) by a single bounded torque $|u_n| \leq a_n$ applied to the last joint, that is to say, the object may be brought [2], say, from its lowest position of equilibrium to the unstable upper position of equilibrium in a limited time, by as small a torque u_n as desired.

2. SYSTEMS WITH STEADY MOTIONS

The sufficient conditions for global controllability used in Proposition 1 are applicable to objects "with strong inter-segmental interactions," when partial dissipation (e.g. owing to friction in one of the joints) may stabilize the entire system, i.e. bring it to as small a neighbourhood of the set ζ_0 of rest states as desired.

In the examples considered below, the sufficient conditions will not be satisfied (in part 1), but the systems will be globally controllable. Their special feature is the presence of steady motions when the system rotates around a stationary axis as a rigid body. Due to the fact that such a state can be approached asymptotically and that the system is locally controllable in the neighbourhood of that state with the available controls, the object can be steered from any initial state to any desired position relative to the rest state, given the value of the angular velocity and the angle of rotation of the whole object. In view of the symmetry with respect to time inversion ($t \to -t$), this implies global controllability.

Example 2. A system with two degrees of freedom (Fig. 2) is situated in a revolving vertical plane in a gravitational field. Weightless rods of length l_1 and l_2 are joined together by cylindrical frictionless joints. Masses m_1 and m_2 are concentrated at the joints. The bounded control $|u_1| \leq a$ is an external torque applied to the first rod. The angle φ is measured in the horizontal plane and the angle ψ is measured from the vertical.

The system does not satisfy the conditions of Proposition 1, since particular solutions $\varphi^{\cdot} \equiv 0$ with $\psi^{\cdot} \not\equiv 0$ exist.

Using dimensionless parameters $\varkappa = m_1/m_2$, $\mu = l_1/l_2$, $u = u_1/(m_2 g l_2)$ and time $\tau = t\sqrt{(g/l_2)}$, we obtain the reduced Lagrangian

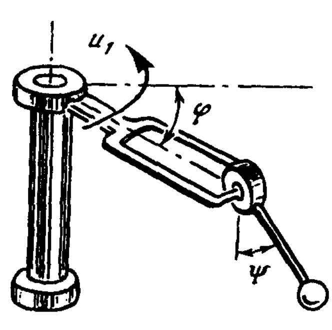

Fig. 2.

$$L=\tfrac{1}{2}\varphi^{\cdot 2}[\varkappa\mu^2+(\mu+\sin\psi)^2]+\tfrac{1}{2}\psi^{\cdot 2}-B(\psi),\quad B(\psi)=1-\cos\psi$$

and the equations of motion are

$$\varphi^{\cdot\cdot}[\varkappa\mu^2+(\mu+\sin\psi)^2]+2\varphi^{\cdot}\psi^{\cdot}\cos\psi(\mu+\sin\psi)=u$$

$$\psi^{\cdot\cdot}-\varphi^{\cdot 2}\cos\psi(\mu+\sin\psi)+\sin\psi=0$$

When $u\equiv 0$, the system admits of a steady rotation $\varphi^{\cdot}=\omega$, $\psi^{\cdot}=0$, $\psi=\psi_0$, with the quantities ω and ψ_0 related by the formula

$$\omega^2=\sin\psi_0/[\cos\psi_0(\mu+\sin\psi_0)] \tag{2.1}$$

We introduce variables $\gamma=\varphi-\omega t$, $\beta=\psi-\psi_0$ and a vector $\mathbf{x}=(\gamma^{\cdot},\beta,\psi^{\cdot})^T$. In the linear approximation in the neighbourhood of the state $\mathbf{x}=0$, the system is

$$\mathbf{x}^{\cdot}=\mathbf{G}\mathbf{x}+\mathbf{b}u,\quad \mathbf{G}=\begin{Vmatrix}0 & 0 & -c_2/c_1\\ 0 & 0 & 1\\ c_2 & -c_3 & 0\end{Vmatrix},\quad \mathbf{b}=\begin{Vmatrix}1/c_1\\ 0\\ 0\end{Vmatrix}$$

$$c_1=\varkappa\mu^2+(\mu+\sin\psi_0)^2>0,\quad c_2=2\omega\cos\psi_0(\mu+\sin\psi_0)\neq 0$$

$$c_3=\omega^2(\mu+\sin^3\psi_0)/\sin\psi_0$$

Since $\det\|\mathbf{b},\mathbf{Gb},\mathbf{G}^2\mathbf{b}\|=-c_2^2/c_1^3\neq 0$, the system is locally controllable by the input u in the neighbourhood of the relative position of rest $\varphi^{\cdot}=\omega$, $\psi^{\cdot}=0$, $\psi=\psi_0$.

For this value of ω, Eq. (2.1) has two roots, one of which satisfies the condition $0<\psi_0<\pi/2$.

We shall show that the system can be brought from any initial state to the ε-neighbourhood of the manifold $\varphi^{\cdot}=\omega$, $\psi=\psi_0$, $\psi^{\cdot}=0$ in a finite time.

Using the artificial potential

$$B_1(\psi)=1-\cos\psi-\tfrac{1}{2}\omega^2[\varkappa\mu^2+(\mu+\sin\psi)^2]$$

we obtain a Lyapunov function

$$V(x)=\tfrac{1}{2}\gamma^{\cdot 2}\{\varkappa\mu^2+[\mu+\sin(\beta+\psi_0)]^2\}+\tfrac{1}{2}\psi^{\cdot 2}+B_1(\beta+\psi_0)-B_1(\psi_0)$$

whose derivative along a trajectory is $V^{\cdot}=u\gamma^{\cdot}$. Choosing an admissible control u to satisfy the condition sign u = $-$sign $\gamma^{\cdot}$, we obtain $V^{\cdot}\leqslant 0$, $V^{\cdot}\neq 0$ (excluding states of relative rest). By Proposition 1, this implies that the system is stabilizable by input u on the manifold $TM_2\backslash\Omega(u)$, where $x\in TM_2$, since $V(x)$ is a CLF on TM_2.

Possible "jamming" of the system about unstable states of relative rest (on $\Omega(u)$) may be overcome due to local controllability, which also guarantees that the state $\mathbf{x}=\mathbf{0}$ will be reached in a finite time.

Thus, this two-segmented system may be steered from any initial condition $(\varphi_1,\varphi^{\cdot}_1,\psi_1,\psi^{\cdot}_1)$ to a state of steady rotation $\varphi^{\cdot}\equiv\omega$, $\psi\equiv\psi_0$, $\psi^{\cdot}\equiv 0$. Since the angle $\varphi(t)$ varies uniformly in such a state, the system will periodically reach, say, the state $(0,\omega,\psi_0,0)$.

We shall show that the system can be steered from this state to the point $(0,0,0,0)$ by an admissible control.

Indeed, by changing the "purposeful" direction of rotation ω, the system could have been steered from any initial position to the state $(0,-\omega,\psi_0,0)$, including from the point $(0,0,0,0)$. Due to the symmetry of system (1.1) relative to time inversion $t\to -t$, the existence of the controllable process $u(t)$: $(0,0,0,0)\to(0,-\omega,\psi_0,0)$ (for $t\in[0,T]$) guarantees the existence of a "symmetrical" trajectory of motion $u(T-t)$: $(0,\omega,\psi_0,0)\to(0,0,0,0)$. This means that there is a possible stepwise process $(\varphi_1,\varphi^{\cdot}_1,\psi_1,\psi^{\cdot}_1)\to(0,\omega,\psi_0,0)\to(0,0,0,0)$ from any initial position.

Thus, in view of the remark at the end of Section 1, the system of Fig. 2 is globally controllable through the action of a single torque u_1, bounded by any previously given quantity a.

3. THE CASE OF n DEGREES OF FREEDOM

We will now extend the logic of the previous arguments to a more general case.

Let us assume that the coordinate $q_1=\varphi$ in the system of rigid bodies (1.1) is cyclic, that is, $\mathbf{q}=(\varphi,\boldsymbol{\psi}^T)^T$, $\boldsymbol{\psi}\in M_1=T^{(r-1)}\times R^{(n-r)}$, $L=1/2\,\mathbf{q}^{\cdot T}\mathbf{A}(\boldsymbol{\psi})\mathbf{q}^{\cdot}-B(\boldsymbol{\psi})$ and also that $\mathbf{u}=(u_1,0,\ldots,0)^T$.

Suppose that the system admits of a motion $\mathbf{u}=\mathbf{0}$, $\varphi^{\cdot}=\omega$, $\boldsymbol{\psi}^{\cdot}=\mathbf{0}$, that is, rotation as a single rigid body. The configuration ψ in a position of relative rest makes the reduced potential

$$B_1(\boldsymbol{\psi})=B(\boldsymbol{\psi})-\tfrac{1}{2}\omega^2a_{11}(\boldsymbol{\psi})+c \tag{3.1}$$

(allowing for translational inertial forces) take an extremal value. The element a_{11} of the matrix $A(\psi)$ determines the moment of inertia of the entire "rigid" system about the axis of rotation.

Let us consider the case in which the function $B_1(\psi)$ has a unique isolated minimum at a fixed value of ω. One can then choose a constant c and assume that the vector ψ is such that $B_1(\boldsymbol{\psi}) \geq 0$, $B_1(0) = 0$.

Define a vector

$$\mathbf{x} = (\gamma^{\cdot},\ \boldsymbol{\psi}^T,\ \boldsymbol{\psi}^{\cdot T})^T \in R \times TM_1, \quad \gamma = \varphi - \omega t$$

The manifold $R \times TM_1$ contains a non-empty set of states of relative rest

$$\zeta_1 = \{\mathbf{x}\colon\ \gamma^{\cdot} = 0,\ \ \partial B_1 / \partial \boldsymbol{\psi} = \mathbf{0},\ \ \boldsymbol{\psi}^{\cdot} = \mathbf{0},\ \ \mathbf{u} = \mathbf{0}\}$$

The total energy may be expressed as

$$E(\mathbf{q}, \mathbf{q}^{\cdot}) = T(\mathbf{x}) + \omega p - \tfrac{1}{2}\omega^2 a_{11}(\boldsymbol{\psi}) + B(\boldsymbol{\psi})$$

$$p = \sum_{i=1}^{n} a_{1i} q_i^{\cdot}, \quad T(\mathbf{x}) = \tfrac{1}{2}(\gamma^{\cdot}, \boldsymbol{\psi}^{\cdot T})\mathrm{A}(\boldsymbol{\psi})(\gamma^{\cdot},\ \boldsymbol{\psi}^{\cdot T})^T$$

where p is the angular momentum of the system of bodies about its axis of rotation and $T(\mathbf{x})$ is a positive-definite form. For the derivatives along trajectories of the vector field $E^{\cdot} = \mathbf{q}^{\cdot T}\mathbf{u}, p^{\cdot} = u_1$, and we therefore obtain for the CLF $V(\mathbf{x}) = T(\mathbf{x}) + B_1(\boldsymbol{\psi})$ the equality $V^{\cdot} = \gamma^{\cdot} u_1$. Thus the conditions of Proposition 1 are satisfied, but not for $R \times TM_1$.

Due to stabilizability by input u_1 in relative motion and local controllability in all neighbourhoods of states of relative rest, the point $\mathbf{x} = \mathbf{0}$ is reachable in a finite time. As a result, the system can be steered from any initial state $(\varphi_1, \varphi_1^{\cdot}, \boldsymbol{\psi}_1^T, \boldsymbol{\psi}_1^{\cdot T})$ to a state of steady rotation $\varphi^{\cdot} \equiv \omega$, where $\boldsymbol{\psi} \equiv \mathbf{0}$, $\boldsymbol{\psi}^{\cdot} \equiv \mathbf{0}$ and the value of the angle φ vanishes periodically.

Hence it follows (by reasoning analogous to that in Example 2) that a controllable process $u_1(t)$: $(0, 0, \mathbf{0}, \mathbf{0}) \to (0, -\omega, \mathbf{0}, \mathbf{0})$ (for $t \in [0, T]$) exists, and hence also a "symmetrical" motion $u_1(T - t)$: $(0, \omega, \mathbf{0}, \mathbf{0}) \to (0, 0, \mathbf{0}, \mathbf{0})$ (for $t \in [0, T]$). Thus, a stepwise transition from any initial state $(\varphi_1, \varphi_1^{\cdot}, \boldsymbol{\psi}_1^T, \boldsymbol{\psi}_1^{\cdot T}) \to (0, \omega, \mathbf{0}, \mathbf{0}) \to (0, 0, \mathbf{0}, \mathbf{0})$ exists. By the Remark at the end of Section 1, this guarantees that system (1.1) is globally controllable by a single torque u_1. We have thus proved the following proposition.

Proposition 2. Suppose that system (1.1) admits of a motion $\mathbf{u} \equiv \mathbf{0}$, $\varphi^{\cdot} = \omega$, $\boldsymbol{\psi}^{\cdot} = \mathbf{0}$, where $\mathbf{q} = (\varphi, \boldsymbol{\psi}^T)^T$, $\boldsymbol{\psi} \in M_1$. Assume in addition that the reduced potential $B_1(\psi)$ in the form (3.1) is a CLF on M_1 and that the sets $H_c(B_1(\psi))$ are compact. Then

1. if in the relative motion $\mathbf{x} = (\gamma^{\cdot}, \boldsymbol{\psi}^T, \boldsymbol{\psi}^{\cdot T})^T$ (where $\mathbf{x} \in R \times TM_1$, $\gamma^{\cdot} - \omega$) with $\mathbf{u} = \mathbf{0}$ there are no particular solutions $\gamma^{\cdot} \equiv 0$ in system (1.1) (except for states of relative rest $\mathbf{x} \in \zeta_1$), then the system is stabilizable by input u_1 in the domain $R \times TM_1 \backslash \Omega(u_1)$;

2. if, in addition, in the neighbourhoods of states $\mathbf{x} \in \zeta_1$ system (1.1) is locally controllable by input u_1, then the system is globally controllable by input u_1 on TM_1.

Example 3. Consider an n-segment pendulum (Fig. 3) in a horizontal plane. This example differs from Example 1 in that now the system is "outside the gravitational field". Again, there is no friction, and a single bounded external torque $|u_1| \leq a$ is applied to the first segment. Here $B(\mathbf{q}) \equiv 0$, and therefore the mechanism may be in equilibrium in any configuration $\mathbf{q} = (\varphi_1, \varphi_2, \ldots, \varphi_n)^T$, where φ_i are the angles between the rods and a fixed axis. The potential energy of the system clearly does not satisfy the conditions of Proposition 1.

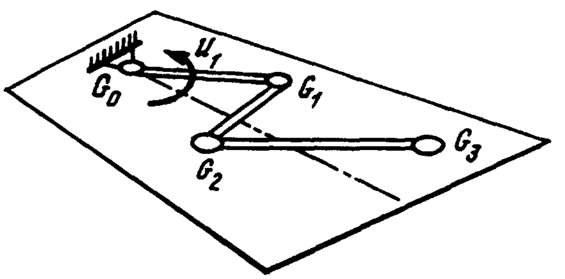

Fig. 3.

Let us assume that the weightless segments are of lengths l_1 ($i = 1, 2, \ldots, n$), and that their masses m_i are concentrated at the joints. Then, in the "straightened" configuration, ($\varphi_i = 0, i = 1, 2, \ldots, n$), we obtain an inertia matrix $\mathbf{A}(\mathbf{0}) = \mathbf{S}$ with elements $s_{ij} = (l_i l_j \sum_{k=r}^{n} m_k)$, $r = \max(i, j)$ $(i, j = 1, 2, \ldots, n)$.

To continue the analysis, it is convenient to express the matrix S in the form $\mathbf{S} = \mathbf{G}(\mathbf{C}^{-1})^T\mathbf{N}\mathbf{C}^{-1}\mathbf{G}$, where $\mathbf{G} = \operatorname{diag}(l_i)$, $\mathbf{N} = \operatorname{diag}(m_i)$ $(i = 1, 2, \ldots, n)$, and the auxiliary matrices are

$$\mathbf{C} = \begin{Vmatrix} 1 & & & & 0 \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & \cdot & \cdot & \cdot \\ 0 & & & -1 & 1 \end{Vmatrix}, \quad \mathbf{C}^{-1} = \begin{Vmatrix} 1 & & & & 0 \\ 1 & 1 & & & \\ \cdot & & \cdot & & \\ \cdot & & & \cdot & \\ 1 & 1 & \cdot & \cdot & 1 \end{Vmatrix}$$

The truth of this decomposition may be verified directly.

The inverse $\mathbf{S}^{-1} = \mathbf{G}^{-1}\mathbf{C}\mathbf{N}^{-1}\mathbf{C}^T\mathbf{G}^{-1}$ is a tridiagonal matrix

$$\mathbf{S}^{-1} = \begin{Vmatrix} \mu_1\lambda_1^2 & -\mu_1\lambda_1\lambda_2 & & & \mathbf{0} \\ -\mu_1\lambda_1\lambda_2 & (\mu_1+\mu_2)\lambda_2^2 & -\mu_2\lambda_2\lambda_3 & & \\ & -\mu_2\lambda_2\lambda_3 & \ddots & & \\ & & \ddots & & -\mu_{n-1}\lambda_{n-1}\lambda_n \\ \mathbf{0} & & & -\mu_{n-1}\lambda_{n-1}\lambda_n & (\mu_{n-1}+\mu_n)\lambda_n^2 \end{Vmatrix}$$

$\mu_i = 1/m_i$, $\lambda_i = 1/l_i$ $(i = 1, 2, \ldots, n)$

We now make the coordinate $q_1 = \varphi_1$ cyclic by transforming from "absolute" angles φ_1 (measured from the stationary axis) to the "relative" angles between the rods: $q_i = \varphi_i - \varphi_{i-1}$ $(i = 2, 3, \ldots, n)$. The relation between the configuration vectors is given by the formula $\mathbf{q} = \mathbf{C}\boldsymbol{\varphi}$.

Put $\boldsymbol{\psi} = (q_2, \ldots, q_n)^T \in M_1$. Then $\mathbf{q} = (\varphi_1, \boldsymbol{\psi}^T)^T$. The Lagrangian $L = \tfrac{1}{2}\mathbf{q}^{\cdot T}\mathbf{A}_1(\boldsymbol{\psi})\mathbf{q}^{\cdot} = \{s_{ij}\cos(\varphi_j - \varphi_i)\}$ is defined by the inertia matrix $\mathbf{A}_1(\psi)$, which is related to the matrix for "absolute angles" $\mathbf{A}(\varphi) = \{s_{ij}\cos(\varphi_j - \varphi_i)\}$ by the condition $\mathbf{A}_1 = (\mathbf{C}^{-1})^T\mathbf{A}\mathbf{C}^{-1}$. As a result the corner element $a_{11}(\psi)$ equals the sum of all elements of the matrix $\mathbf{A}(\varphi)$ (with the substitution $\varphi = \mathbf{C}^{-1}\mathbf{q}$).

A steady motion $u_1 = 0$, $\varphi_1^{\cdot} = \omega$, $\boldsymbol{\psi}^{\cdot} = \mathbf{0}$ is possible if $\partial B_1/\partial\boldsymbol{\psi} = \mathbf{0}$, $B_1(\boldsymbol{\psi}) = 1/2\omega^2[a_{11}(\mathbf{0}) - a_{11}(\boldsymbol{\psi})]$. The reduced potential $B_1(\psi)$ (translational forces of inertia) has a single isolated minimum at the point $\psi = \mathbf{0}$. It follows from the sufficient condition for connectedness of the Lyapunov function [5] (Section 1) that $B_1(\psi)$ is a *CLF* on M_1.

We now introduce the vector $\mathbf{x} = (\gamma, \boldsymbol{\psi}^T, \boldsymbol{\psi}^{\cdot T})^T \in R \times TM$, where $\gamma = \varphi - \omega t$. The manifold $R \times TM_1$ contains a non-empty set $\zeta_1 = \{\mathbf{x}: \gamma = 0, \partial B_1/\partial\boldsymbol{\psi} = \mathbf{0}, \boldsymbol{\psi}^{\cdot} = \mathbf{0}, u = \mathbf{0}\}$ of states of relative rest.

Let us investigate the conditions for local controllability by a single torque in the neighbourhoods of points $\mathbf{x} \in \zeta_1$. The system, linearized in the neighbourhood of $\mathbf{x} = \mathbf{0}$ (i.e. $\varphi^{\cdot} = \omega$, $\psi = 0$, $\boldsymbol{\psi}^{\cdot} = \mathbf{0}$), has the form $\mathbf{A}_0\mathbf{q}^{\cdot\cdot} = \mathbf{B}_0\mathbf{q} + \mathbf{b}u_1$, $\mathbf{b} = (1, 0, \ldots, 0)^T$. In the computations of $\mathbf{A}_0 = (\mathbf{C}^{-1})^T\mathbf{S}\mathbf{C}^{-1}$, $\mathbf{B}_0 = (\mathbf{C}^{-1})^T\mathbf{H}\mathbf{C}^{-1}$ we have used the matrix

$$\mathbf{H} = \left\{\omega^2\frac{\partial^2}{\partial\varphi^2}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}s_{ij}\cos(\varphi_j - \varphi_i)\right]\right\}\Bigg|_{\varphi=\mathbf{0}}$$

It can be verified that $\mathbf{H} = \omega^2(\mathbf{S} - \mathbf{F})$, where $\mathbf{F} = \operatorname{diag}(f_i)$, $(i = 1, 2, \ldots, n)$, with $f_i = \Sigma s_{ij}$ $(j = 1, 2, \ldots, n)$. Changing to new variables $\mathbf{Z} = \mathbf{S}\mathbf{C}^{-1}\mathbf{q}$ and taking into account that $\mathbf{C}^T\mathbf{b} = \mathbf{B}$, we obtain $\mathbf{Z}^{\cdot\cdot} = \mathbf{D}\mathbf{z} + \mathbf{b}u_1$.

It is well known [7] that in such a system with non-singular matrix $\mathbf{D}$ the rank criterion for controllability reduces to the form rank $\mathbf{K} = n$, where $\mathbf{K} = \|\mathbf{b}, \mathbf{D}\mathbf{b}, \ldots, \mathbf{D}^{n-1}\mathbf{b}\|$.

In the present example, $\mathbf{D} = \omega^2(\mathbf{I} - \mathbf{F}\mathbf{S}^{-1})$ is tridiagonal, but in the triangular controllability matrix $\mathbf{K}$ we have the diagonal elements $k_{11} = 1$, $k_{i+1,i+1} = -k_{ii}\omega^2 f_{i+1}\mu_i\lambda_{i+1}\lambda_i \neq 0$ $(i = 1, 2, \ldots, n-1)$, so that rank $\mathbf{K} = n$.

This means that in the neighbourhood of the point $\mathbf{x} = \mathbf{0}$ in $R \times TM_1$ the system is locally controllable by the action of a single torque u_1.

In the neighbourhoods of other positions $\mathbf{x} \in \zeta_1$ of relative equilibrium, analogous arguments lead to the following conclusion for an n-segment pendulum outside the gravitational field: if in the inertia matrix $\mathbf{A}(\varphi) = \{a_{ij}\}$ in the rectilinear configuration (when $a_{ij} = \pm s_{ij}$) the values of all row sums $f_i = \Sigma a_{ij}$ $(j = 1, 2, \ldots, n)$ do not vanish (for all $i = 2, 3, \ldots, n$), the system is locally controllable in the neighbourhood of the appropriate position of relative equilibrium ($\omega \neq 0$) under the action of a single torque u_1.

To simplify the discussion, let us consider the case $n = 3$. We will discuss conditions under which a three-segment system admits of no particular solution $\varphi_1^{\cdot} = \omega$ (excluding states $\mathbf{x} \in \zeta_1$ of relative rest). Denote the point of suspension of the pendulum by G_0 and the ends of the rods (the joints) by G_1, G_2, G_3, respectively (Fig. 3).

Suppose that the segment G_0G_1 is at relative rest when $\varphi_1^{\cdot} = \omega$. Then the second rod exerts a reaction force on the first along G_0G_1. Taken separately, the segment G_1G_2 can be at equilibrium under the reactions of the constraints and the d'Alembert forces only if $N_2 \sin \psi_1 = 0$, where N_2 is the pressure force exerted by the rod G_1G_2. The condition $N_2 \equiv 0$ is equivalent to violation of the constraint, when the third segment moves freely, i.e. rotates about the stationary centre of mass or is at rest in an absolute system of coordinates. This may happen only if the points G_0 and G_2 are geometrically identical.

Thus, apart from the case of states of relative equilibrium a particular integral $\varphi_1^{\cdot} = \omega$ is possible if $\psi_1 = \pi$, $\psi^{\cdot} = 0$, $\psi_2^{\cdot} = -\omega$, when the third segment is at rest and the first two rotate in a combined configuration (the points G_2 and G_0 in Fig. 3 coincide).

Analysis of the equations of a two-segment pendulum with equal rods ($n = 2$, $l_1 = l_2$) shows that the manifold $\psi_1 = \pi$, $\psi^{\cdot}_1 = 0$ is invariant with respect to u_1, i.e. rotation of superimposed rods cannot be produced by any $u_1(t)$. This property cannot be neutralized by adding a third segment G_2G_3 at rest to the moving system, since the points G_0 and G_2 are superimposed.

Thus, the three-segment pendulum ($l_1 = l_2$) admits of an invariant manifold $\Gamma = \{(\mathbf{q}, \mathbf{q}^{\cdot}): \psi_1 = \pi, \psi_1^{\cdot} = 0, \psi_2^{\cdot} = -\varphi_1^{\cdot}\}$. The domain $TM \backslash \Gamma$ is open and everywhere dense in TM.

From any initial point $\mathbf{q}_1$, $\mathbf{q}_1^{\cdot} \in TM\backslash\Gamma$, the system can be steered by a stabilizing control sign $u_1 = -\text{sign}\,(\varphi^{\cdot}_1 - \omega)$ into the neighbourhood of points $\mathbf{x} \in \zeta_1$ in a finite time, since the system has no particular solutions $\gamma = 0$ other than points $\mathbf{x} \in \zeta_1$.

Using the logic of Proposition 2, we conclude that the domain of controllability of the pendulum under consideration ($n = 3$, $l_1 = l_2$) under the action of a single torque u_1 is TM/Γ.

By suitable choice of the quantities m_i, l_i ($i = 1, 2, \ldots, n$), one can completely avoid the existence of invariant manifolds and guarantee global controllability of a plane multi-segment pendulum outside the gravitational field by applying a single, arbitrarily small torque.

We note, finally, that a physical n-segment pendulum outside the gravitational field does not admit of an invariant manifold as described if the centre of gravity of the second segment does not coincide with the point of suspension.

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